

Duality of translation association schemes coming from certain actions

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Abstract

Translation association schemes are constructed from actions of finite groups on finite abelian groups satisfying certain natural conditions. It is also shown that the mere existence of maps from finite groups to themselves sending each element in their groups to its ‘adjoint’ entails the self-duality of the constructed association schemes. Many examples of these, including Hamming scheme and sesquilinear forms schemes, are provided. This construction is further generalized to show the duality of the association schemes coming from actions of two finite groups on the same finite abelian group. An example of this is supplied with weak Hamming schemes.

Key words: Action ; Orbit ; Translation association scheme ; Adjoint ;
Self-duality ; Duality

1 Introduction

Let G be a finite group acting on a finite additive abelian group X , and let $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_1, \dots, \mathcal{O}_d$ be the G -orbits. Assume that the action satisfies the conditions

$$g(x + x') = gx + gx', \text{ for all } g \in G \text{ and } x, x' \in X,$$

and

$$x \in \mathcal{O}_i \Rightarrow -x \in \mathcal{O}_i, \text{ for } 0 \leq i \leq d.$$

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This work was supported by grant No. R01-2007-000-11176-0 from the Basic Research Program of the Korea Science and Engineering Foundation.

Then it is easy to see that $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$, with $(x, y) \in R_i \Leftrightarrow y - x \in \mathcal{O}_i$, is a translation association scheme (cf. Theorem 2).

Assume now further that the following map sending each element to its adjoint exists, i.e., there is a map $\iota : G \rightarrow G$ such that

$$\langle gx, y \rangle = \langle x, \iota(g)y \rangle, \text{ for all } g \in G, x, y \in X.$$

Here $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}^\times$ is an inner product on X . Then it is shown that such a simple requirement is enough to guarantee the self-duality of \mathfrak{X}_G (cf. Corollary 7). Many examples of such pairs (G, X) satisfying the above three conditions are provided. For instance, the Hamming scheme and most of sesquilinear forms schemes fall within this category (cf. Section 4).

Suppose now that \tilde{G} is another finite group acting on the same finite abelian group X , with $\tilde{\mathcal{O}}_0 = \{0\}$, $\tilde{\mathcal{O}}_1, \dots, \tilde{\mathcal{O}}_d$ the \tilde{G} -orbits, and that the corresponding conditions to (1) and (2) for \tilde{G} and $\tilde{\mathcal{O}}_i$ ($0 \leq i \leq d$) are satisfied. Assume in addition that there is a map $\iota : G \rightarrow \tilde{G}$ such that

$$\langle gx, y \rangle = \langle x, \iota(g)y \rangle, \text{ for all } g \in G, x, y \in X.$$

Then it is shown that the existence of the map sending each element to its adjoint is strong enough to yield the duality of \mathfrak{X}_G and $\mathfrak{X}_{\tilde{G}}$. An example of this is illustrated with what we call the weak Hamming scheme $H(n_1, \dots, n_t, q)$. It is the wreath product of the Hamming schemes $H(n_1, q), \dots, H(n_t, q)$. Our result now says that $H(n_1, \dots, n_t, q)$ and $H(n_t, \dots, n_1, q)$ are dual to each other. This would have been observed already in [14] or even in the earlier works [10] and [13]. The weak Hamming scheme can be also constructed in connection with weak order poset weight (cf. Section 6). Also, it is an example of weak metric schemes which are the subject of the recent paper [12].

2 Preliminaries

A pair $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ consisting of a finite set X (referred to as the vertex set of \mathfrak{X}) and $d + 1$ nonempty subsets R_i of $X \times X$ is called a d -class (symmetric) association scheme if

- (i) $\{R_0, R_1, \dots, R_d\}$ is a partition of $X \times X$,
- (ii) $R_0 = \Delta_X = \{(x, x) | x \in X\}$,
- (iii) ${}^t R_i = R_i$, for all i , where ${}^t R_i = \{(x, y) | (y, x) \in R_i\}$,
- (iv) for any i, j, k ($0 \leq i, j, k \leq d$), there are numbers, called intersection numbers, p_{ij}^k such that, for any $(x, y) \in R_k$,

$$p_{ij}^k = \#\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}.$$

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be an association scheme. Let A_i be the adjacency matrix of R_i , for $0 \leq i \leq d$. Then A_0, A_1, \dots, A_d generate a $(d+1)$ -dimensional commutative subalgebra \mathcal{A} of symmetric matrices in $M_n(\mathbb{C})$ ($n = |X|$), called the Bose-Mesner algebra of \mathfrak{X} . \mathcal{A} has another nice basis E_0, E_1, \dots, E_d , called the irreducible idempotents of \mathcal{A} . They are determined (up to permutation of the indices $1, \dots, d$) by :

$$\begin{aligned} & \text{(i) } E_i E_j = \delta_{ij} E_i, \text{ for all } i, j, \\ & \text{(ii) } \sum_{i=0}^d E_i = I, \\ & \text{(iii) } E_0 = |X|^{-1} J \text{ (} J \text{ all-one matrix),} \\ & \text{(iv) } E_0, E_1, \dots, E_d \text{ are linearly independent over } \mathbb{C}. \end{aligned} \tag{1}$$

The \mathbb{C} -space generated by A_0, A_1, \dots, A_d is also closed under the Hadamard multiplication \circ , with J as the multiplicative identity. Write

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^d q_{ij}^k E_k \quad (0 \leq i, j \leq d).$$

Then q_{ij}^k 's are actually nonnegative real numbers, called the Krein parameters. Let

$$A_j = \sum_{i=0}^d p_{ij} E_i, \quad E_j = |X|^{-1} \sum_{i=0}^d q_{ij} A_i.$$

The p_{ij} 's and q_{ij} 's are respectively called p - and q -numbers. In particular, $v_i = p_{0i} = \#\{z | (x, z) \in R_i\}$, for any fixed $x \in X$, and $m_i = q_{0i} = \text{rank } E_i$ are respectively called the valencies and the multiplicities. Also, $P = (p_{ij})$ and $Q = (q_{ij})$ are respectively called the first and the second eigenmatrix of \mathfrak{X} .

Let X be a finite additive abelian group, and let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a d -class association scheme. Then \mathfrak{X} is called a translation association scheme if

$$(x, y) \in R_i \Rightarrow (x + z, y + z) \in R_i, \text{ for all } z \in X \text{ and } i.$$

Let

$$X_i = \{x \in X | (0, x) \in R_i\}, \text{ for } 0 \leq i \leq d. \tag{2}$$

Then X_0, X_1, \dots, X_d give a partition of X , and

$$(x, y) \in R_i \Leftrightarrow y - x \in X_i \quad (0 \leq i \leq d).$$

The dual association scheme $\mathfrak{X}^* = (X^*, \{R_i^*\}_{i=0}^d)$ of \mathfrak{X} consists of the group X^* of characters on X together with $d+1$ nonempty subsets R_i^* of $X^* \times X^*$ determined by :

$$(\chi, \psi) \in R_i^* \Leftrightarrow \psi \chi^{-1} \in X_i,$$

where $X_i^* = \{\chi \in X^* | E_i \chi = \chi\}$. Here χ is viewed as the column vector with the x -component ($x \in X$) given by $\chi(x)$. For the proof of the following theorem, see [3, Theorem 2.10.10].

Theorem 1 *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a translation association scheme with parameters $p_{ij}^k, q_{ij}^k, v_i, m_i, P, Q$. Then we have the following :*

- (a) *The dual scheme $\mathfrak{X}^* = (X, \{R_i^*\}_{i=0}^d)$ is also a translation association scheme with parameters $p_{ij}^{*k} = q_{ij}^k, q_{ij}^{*k} = p_{ij}^k, v_i^* = m_i, m_i^* = v_i, P^* = Q, Q^* = P$.*
- (b) (i) $p_{ij} = \sum_{x \in X_j} \chi(x)$, for $\chi \in X_i^*$,
(ii) $q_{ij} = \sum_{\chi \in X_j^*} \chi(x)$, for $x \in X_i$,
(iii) $E_j = |X|^{-1} \sum_{\chi \in X_j^*} \chi \ ^t \overline{\chi}$,
(iv) $v_j = |X_j|$,
(v) $m_j = |X_j^*|$.

Two association schemes $\mathfrak{X} = (X, \{R_{X,i}\}_{i=0}^d)$ and $\mathfrak{Y} = (Y, \{R_{Y,i}\}_{i=0}^d)$ are said to be isomorphic if there are a bijection $f : X \rightarrow Y$ and a permutation σ of $\{1, 2, \dots, d\}$ such that

$$(x, y) \in R_{X,i} \Leftrightarrow (f(x), f(y)) \in R_{Y,\sigma(i)}, \text{ for } 1 \leq i \leq d.$$

Here we always assume that $R_{X,0} = \Delta_X, R_{Y,0} = \Delta_Y$, so that $(x, y) \in R_{X,0} \Leftrightarrow (f(x), f(y)) \in R_{Y,0}$. In particular, two d -class translation association schemes $\mathfrak{X} = (X, \{R_{X,i}\}_{i=0}^d)$ and $\mathfrak{Y} = (Y, \{R_{Y,i}\}_{i=0}^d)$ are isomorphic if there is a group isomorphism $f : X \rightarrow Y$ and a permutation σ of $\{1, 2, \dots, d\}$ such that

$$x \in X_i \Leftrightarrow f(x) \in Y_{\sigma(i)}, \text{ for } 1 \leq i \leq d.$$

Here $X_i = \{x \in X | (0, x) \in R_{X,i}\}$, $Y_i = \{y \in Y | (0, y) \in R_{Y,i}\}$. If two association schemes are isomorphic, we may assume that all the parameters of the schemes are the same.

For further facts about association schemes, one is referred to [2] and [3].

3 Construction of translation association schemes

Let G be a finite group acting on a finite additive abelian group X , with $\mathcal{O}_0 = \{0\}, \mathcal{O}_1, \dots, \mathcal{O}_d$ the G -orbits. We assume that this action satisfies the conditions

$$g(x + x') = gx + gx', \text{ for all } g \in G, x, x' \in X, \quad (3)$$

and

$$x \in \mathcal{O}_i \Rightarrow -x \in \mathcal{O}_i, \text{ for } 0 \leq i \leq d. \quad (4)$$

Note here that $g0 = 0$, for all $g \in G$, as $\mathcal{O}_0 = \{0\}$. This together with (3) implies that

$$g(-x) = -gx, \text{ for all } g \in G, x \in X. \quad (5)$$

A special case of the following theorem appeared in [11].

Theorem 2 $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$, given by

$$(x, y) \in R_i \Leftrightarrow y - x \in \mathcal{O}_i \quad (0 \leq i \leq d),$$

is a translation association scheme.

PROOF. Here the conditions (3) and (4) are needed. All are easy to check, perhaps except for the condition on the intersection numbers. Let $x, y, x', y' \in X$, with $u = y - x$, $v = y' - x' \in \mathcal{O}_k$. Then we must see :

$$\#\{z \in X | z - x \in \mathcal{O}_i, y - z \in \mathcal{O}_j\} = \#\{z \in X | z - x' \in \mathcal{O}_i, y' - z \in \mathcal{O}_j\}.$$

Observe that

$$\{z \in X | z - x \in \mathcal{O}_i, y - z \in \mathcal{O}_j\} \rightarrow \{z \in X | u - z \in \mathcal{O}_i, z \in \mathcal{O}_j\} \quad (z \mapsto y - z)$$

is a bijection. So it is enough to show :

$$\#\{z \in X | u - z \in \mathcal{O}_i, z \in \mathcal{O}_j\} = \#\{z \in X | v - z \in \mathcal{O}_i, z \in \mathcal{O}_j\}.$$

As G acts transitively on each \mathcal{O}_i , and $u, v \in \mathcal{O}_k$, there is an $h \in G$ such that $hu = v$, and hence

$$\{z \in X | u - z \in \mathcal{O}_i, z \in \mathcal{O}_j\} \rightarrow \{z \in X | v - z \in \mathcal{O}_i, z \in \mathcal{O}_j\} \quad (z \mapsto hz)$$

is a bijection. Notice that (3), (4) and (5) are used here. \square

Remark 3 (1) For the association scheme $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$,

$$X_i = \mathcal{O}_i, \text{ for } 0 \leq i \leq d \text{ (cf. (2)).}$$

(2) Let the condition (4) be replaced by :

$$\text{for each } i \quad (0 \leq i \leq d), x \in \mathcal{O}_i \Rightarrow -x \in \mathcal{O}_j, \text{ for some } j. \quad (6)$$

If (3) is satisfied along with (6), then in fact we have $x \in \mathcal{O}_i \Leftrightarrow -x \in \mathcal{O}_j$, and hence the association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$, given by $(x, y) \in R_i \Leftrightarrow y - x \in \mathcal{O}_i \quad (0 \leq i \leq d)$, is not a symmetric and yet commutative association scheme. However, in this paper we will consider only the association schemes

coming from the actions satisfying (3) and (4).

Let $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}^\times$ be an inner product on the finite additive abelian group X , i.e.,

- (i) $\langle x, y \rangle = \langle y, x \rangle$, for all $x, y \in X$,
- (ii) $\langle x, y + z \rangle = \langle x, y \rangle \langle x, z \rangle$, for all $x, y, z \in X$,
- (iii) $\langle x, y \rangle = \langle x, z \rangle$, for all $x \in X \Rightarrow y = z$.

Remark 4 *It is well-known that such an inner product always exists on a finite abelian group X . $\langle \cdot, x \rangle$ will denote the character on X given by $y \mapsto \langle y, x \rangle$, so that*

$$X^* = \{ \langle \cdot, x \rangle \mid x \in X \}.$$

Also, $\langle \cdot, x \rangle$ will indicate the column vector of size $|X|$ whose y -component is $\langle y, x \rangle$ ($y \in X$).

Theorem 5 *The following are equivalent.*

- (a) $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d) \rightarrow \mathfrak{X}_G^* = (X^*, \{R_i^*\}_{i=0}^d)$, given by $x \mapsto \langle \cdot, x \rangle$, is an isomorphism of translation association schemes.
- (b) There is a permutation σ of $\{1, 2, \dots, d\}$ such that

$$x \in X_j \Leftrightarrow \langle \cdot, x \rangle \in X_{\sigma(j)}^*, \text{ for } j = 1, \dots, d.$$

- (c) $E_j = |X|^{-1} \sum_{x \in X_j} \langle \cdot, x \rangle \overline{{}^t \langle \cdot, x \rangle}$ ($0 \leq j \leq d$) are the irreducible idempotents of the association scheme $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$.
- (d) $f_j : X \rightarrow \mathbb{C}$, given by $f_j(y) = \sum_{x \in X_j} \langle y, x \rangle$, is constant on each X_i ($0 \leq i \leq d$), for all $j = 0, \dots, d$.

PROOF. (a) \Leftrightarrow (b) This is just the definition.

(a) \Rightarrow (d) As (a) \Leftrightarrow (b), $q_{i\sigma(j)} = \sum_{x \in X_j} \langle y, x \rangle$, for $y \in X_i$, is the q -number of \mathfrak{X} , and hence is constant on each X_i ($0 \leq i \leq d$), for all $j = 0, 1, \dots, d$.

(d) \Rightarrow (c) Let $\overline{E_j} = |X|^{-1} \sum_{x \in X_j} \langle \cdot, x \rangle \overline{{}^t \langle \cdot, x \rangle}$, for all j . Then $\overline{E_j} = |X|^{-1} \sum_{i=0}^d \overline{q_{ij}} A_i$, with $\overline{q_{ij}} = \sum_{x \in X_j} \langle y, x \rangle$ ($y \in X_i$). So $\overline{E_j}$ belongs to the Bose-Mesner algebra of \mathfrak{X}_G .

$$(\overline{E_i} \overline{E_j})_{ab} = |X|^{-2} \sum_{\substack{x \in X_i \\ y \in X_j}} \langle a, x \rangle \overline{\langle b, y \rangle} \overline{{}^t \langle \cdot, x \rangle} \langle \cdot, y \rangle.$$

Note that

$$\overline{{}^t \langle \cdot, x \rangle} \langle \cdot, y \rangle = \sum_{z \in X} \langle z, y - x \rangle = \begin{cases} 0, & y \neq x, \\ |X|, & y = x. \end{cases}$$

So

$$(\overline{E_i E_j})_{ab} = \begin{cases} 0, & i \neq j, \\ |X|^{-1} \sum_{x \in X_i} \langle a - b, x \rangle = (\overline{E_i})_{ab}, & i = j, \end{cases}$$

and hence $\overline{E_i E_j} = \delta_{ij} \overline{E_i}$. Similarly, $\sum_{i=0}^d \overline{E_i} = I$. Assume that $\sum_{i=0}^d \alpha_i \overline{E_i} = 0$, with $\alpha_i \in \mathbb{C}$. Then $\alpha_i \overline{E_i} = 0$, for all i . To show independence of $\overline{E_0}, \dots, \overline{E_d}$, it is enough to see that $\overline{E_i} \neq 0$, for all i . This is indeed the case, since

$$\overline{E_i} \langle \cdot, x \rangle = \begin{cases} \langle \cdot, x \rangle, & \text{if } x \in X_i, \\ 0, & \text{if } x \notin X_i. \end{cases} \quad (7)$$

Thus E_0, E_1, \dots, E_d are the irreducible idempotents of \mathfrak{X} (cf. (1)). Note here that $\overline{E_0} = |X|^{-1} J$.

(c) \Rightarrow (b) Recall that $\tilde{E}_0, \tilde{E}_1, \dots, \tilde{E}_d$, with $\tilde{E}_j = |X|^{-1} \sum_{\chi \in X_j^*} \chi \ ^t \overline{\chi}$, is also the irreducible idempotents of \mathfrak{X}_G (cf. Theorem 1, (b)). Observe also that $\tilde{E}_0 = |X|^{-1} J$. As the irreducible idempotents are unique up to permutation, there is a permutation σ of $\{1, 2, \dots, d\}$ such that $E_j = \tilde{E}_{\sigma(j)}$, for $j = 1, \dots, d$. Now, using (7) we have :

$$\begin{aligned} x \in X_j &\Leftrightarrow E_j \langle \cdot, x \rangle = \langle \cdot, x \rangle \\ &\Leftrightarrow \tilde{E}_{\sigma(j)} \langle \cdot, x \rangle = \langle \cdot, x \rangle \\ &\Leftrightarrow \langle \cdot, x \rangle \in X_{\sigma(j)}^*. \quad \square \end{aligned}$$

Assume now further that there is a map $\iota : G \rightarrow G$ such that

$$\langle gx, y \rangle = \langle x, \iota(g)y \rangle, \text{ for all } g \in G, x, y \in X, \quad (8)$$

where $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}^\times$ is an inner product.

Lemma 6 *Under the assumption of (8), the sum*

$$\sum_{x \in X_j} \langle y, x \rangle \quad (y \in X_i)$$

depends only on i , for all i, j with $0 \leq i, j \leq d$.

PROOF. Let $y_1, y_2 \in X_i$. Then $y_2 = hy_1$, for some $h \in G$. So

$$\begin{aligned} \sum_{x \in X_j} \langle y_2, x \rangle &= \sum_{x \in X_j} \langle hy_1, x \rangle \\ &= \sum_{x \in X_j} \langle y_1, \iota(h)x \rangle \\ &= \sum_{x \in X_j} \langle y_1, x \rangle. \quad \square \end{aligned}$$

Now, we get the following corollary from Theorem 5 which says in particular that \mathfrak{X}_G is self-dual.

Corollary 7 *Let $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$ be the translation association scheme obtained from the action of the finite group G on the finite abelian group X , satisfying (3) and (4). Assume that there is a map $\iota : G \rightarrow G$ such that*

$$\langle gx, y \rangle = \langle x, \iota(g)y \rangle, \text{ for all } g \in G, x, y \in X,$$

where $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}^\times$ is an inner product. Then

- (a) $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d) \rightarrow \mathfrak{X}_G^* = (X^*, \{R_i^*\}_{i=0}^d)$, given by $x \mapsto \langle \cdot, x \rangle$, is an isomorphism, i.e., \mathfrak{X}_G is self-dual,
- (b) $E_j = |X|^{-1} \sum_{x \in X_j} \langle \cdot, x \rangle {}^t \overline{\langle \cdot, x \rangle}$ ($0 \leq j \leq d$) are the irreducible idempotents for \mathfrak{X}_G ,
- (c) $q_{ij} = \sum_{x \in X_j} \langle y, x \rangle$ ($y \in X_i$) are the q -numbers for \mathfrak{X}_G ,
- (d) $p_{ij}^k = q_{ij}^k$, $p_{ij} = q_{ij}$, $v_i = m_i$.

4 Examples for Section 3

Here we will demonstrate that there are abundant examples of actions of finite groups G on finite abelian groups X satisfying (3) and (4), and (8) for suitable inner products on X . So the schemes $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$ constructed from these actions are, in particular, self-dual. In below, λ will always denote a fixed nontrivial additive character on \mathbb{F}_q . The examples (a) and (b) below are adopted from [6].

- (a) Let X be a finite abelian group with period ν . Then $(\mathbb{Z}/(\nu))^\times$ acts on X via

$$(\mathbb{Z}/(\nu))^\times \times X \rightarrow X \ ((\overline{m}, x) \mapsto mx).$$

This action satisfies (3) and (4), and the orbits $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_1, \dots, \mathcal{O}_d$ are called the central classes of X . Further, given any inner product on X , (8) is satisfied with ι the identity map.

- (b) Let ω be a primitive element in \mathbb{F}_q (q an odd prime power), and let d be a positive integer such that $2d \mid q-1$. Let $G = \langle \omega^d \rangle$ be the cyclic subgroup of \mathbb{F}_q^\times of order $r = (q-1)/d$. Then G acts on $X = (\mathbb{F}_q, +)$ by left multiplication. Here the orbits are $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_1, \dots, \mathcal{O}_d$, where, for $0 \leq i \leq d-1$,

$$\mathcal{O}_{i+1} = \{\omega^i, \omega^{d+i}, \dots, \omega^{(r-1)d+i}\}.$$

$\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_d$ are called the cyclotomic classes of \mathbb{F}_q . The condition (3) is obviously satisfied, and the condition (4) is also valid, as we assume $2d \mid q-1$. Then $\langle x, y \rangle = \lambda(xy)$ is an inner product on X , and (8) is satisfied with ι the identity map.

The examples (c)-(f) will be about sesquilinear forms association schemes. There are many articles about this topic. Here we are content with just mentioning [3, Sections 9.5-6] and [5, 7, 8, 16].

(c) Let $X = (\mathbb{F}_q^{m \times n}, +)$, with $m \leq n$, and let $G = GL(m, q) \times GL(n, q)$ be the direct product of general linear groups. G now acts on X via

$$G \times X \rightarrow X \ ((\alpha, \beta), A) \mapsto \phi^{(\alpha, \beta)} A := {}^t \alpha A \beta.$$

Then $\mathcal{O}_i = \{A \in X \mid \text{rank}(A) = i\}$ ($i = 0, 1, \dots, m$) are the G -orbits, and the conditions (3) and (4) hold. The associated scheme \mathfrak{X}_G is called the bilinear forms scheme, which is usually denoted by $Bil(m \times n, q)$. Moreover, for $A \cdot B = \text{tr}(A {}^t B) = \sum_{i,j} A_{ij} B_{ij}$ ($A, B \in X$), and $(\alpha, \beta) \in G$,

$$\phi^{(\alpha, \beta)} A \cdot B = A \cdot \phi^{(t\alpha, t\beta)} B.$$

So, for the inner product $\langle A, B \rangle = \lambda(A \cdot B)$ and $\iota : G \rightarrow G \ ((\alpha, \beta) \mapsto ({}^t \alpha, {}^t \beta))$, (8) is valid.

(d) Let X be the group of all alternating matrices of order m over \mathbb{F}_q . Recall here that (A_{ij}) is alternating $\Leftrightarrow A_{ii} = 0$, for $1 \leq i \leq m$, and $A_{ji} = -A_{ij}$, for $1 \leq i < j \leq m$. $G = GL(m, q)$ acts on X via

$$G \times X \rightarrow X \ ((\alpha, A) \mapsto \phi^{(\alpha)} A := {}^t \alpha A \alpha).$$

Then $\mathcal{O}_i = \{A \in X \mid \text{rank}(A) = 2i\}$ ($i = 0, 1, \dots, n = \lfloor \frac{m}{2} \rfloor$) are the G -orbits, and the conditions (3) and (4) are satisfied. The associated scheme \mathfrak{X}_G is called the alternating forms scheme which is denoted by $Alt(m, q)$. For $A \cdot B = \sum_{i < j} A_{ij} B_{ij}$ ($A, B \in X$), and $\alpha \in G$,

$$\phi^{(\alpha)} A \cdot B = A \cdot \phi^{(t\alpha)} B.$$

This holds regardless of the characteristic of \mathbb{F}_q . But it is easier to check this for $\text{char } \mathbb{F}_q \neq 2$, since $A \cdot B = 2^{-1} \text{tr} A {}^t B$ in that case. So, for the inner product $\langle A, B \rangle = \lambda(A \cdot B)$, and $\iota : G \rightarrow G \ (\alpha \mapsto {}^t \alpha)$, (8) is satisfied.

(e) Let X be the group of all Hermitian matrices of order m over \mathbb{F}_{q^2} . Recall here that $A \in X$ is Hermitian if ${}^* A = A$, with $({}^* A)_{ij} = \overline{A_{ji}} = (A_{ji})^q$. $G =$

$GL(m, q^2)$ acts on X via

$$G \times X \rightarrow X \ ((\alpha, A) \mapsto \phi^{(\alpha)} A = {}^* \alpha A \alpha).$$

Then $\mathcal{O}_i = \{A \in X | \text{rank}(A) = i\}$ ($i = 0, 1, \dots, m$) are the G -orbits, and the conditions (3) and (4) are satisfied. The associated scheme \mathfrak{X}_G is called the Hermitian forms scheme which is denoted by $Her(m, q^2)$. For $A \cdot B = \sum_{i,j} A_{ij} \overline{B_{ij}} = \text{tr}(A {}^* B) = \text{tr}(AB)$ ($A, B \in X$), and $\alpha \in G$,

$$\phi^{(\alpha)} A \cdot B = A \cdot \phi^{({}^* \alpha)} B.$$

Note here that $A \cdot B \in \mathbb{F}_q$, and hence that $\langle A, B \rangle = \lambda(A \cdot B)$ makes sense and $\langle A, B \rangle$ is an inner product on X . Now, (8) holds for $\iota : G \rightarrow G$ ($\alpha \mapsto {}^* \alpha$).

(f) Let X be the group of all symmetric matrices of order m over \mathbb{F}_q . Let q be odd. $G = GL(m, q)$ acts on X via

$$G \times X \rightarrow X \ ((\alpha, A) \mapsto \phi^{(\alpha)} A := {}^t \alpha A \alpha).$$

Then $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_{r,+} = \{A \in X | A \sim J_r^+\}$, $\mathcal{O}_{r,-} = \{A \in X | A \sim J_r^-\}$ ($r = 1, \dots, m$) are the G -orbits, where $J_r^+ = I_r \dot{+} O$, $J_r^- = \varepsilon I_1 \dot{+} I_{r-1} \dot{+} O$, with ε a fixed nonsquare element in \mathbb{F}_q . Here ‘ \sim ’ and ‘ $\dot{+}$ ’ indicate respectively ‘cogredient’ and the matrix direct sum. Also, for this well-known fact one is referred to [15, Chap.IV]. The condition (3) is clearly satisfied. Assume further that $q \equiv 1 \pmod{4}$. Then, as -1 is a square, (4) is also valid. The associated scheme \mathfrak{X}_G is called the symmetric forms scheme. For $A \cdot B = \sum_{i,j} A_{ij} B_{ij} = \text{tr}(A {}^t B) = \text{tr}(AB)$ ($A, B \in X$), and $\alpha \in G$,

$$\phi^{(\alpha)} A \cdot B = A \cdot \phi^{({}^t \alpha)} B.$$

Observe here that $\langle A, B \rangle = \lambda(A \cdot B)$ is indeed an inner product on X , since we assume q is odd. Now, (8) holds for $\iota : G \rightarrow G$ ($\alpha \mapsto {}^t \alpha$). On the other hand, if $q \equiv 3 \pmod{4}$, then (4) is not satisfied but (6) is. So in that case the associated scheme is a commutative association scheme (cf. Remarks 3).

(g) Let $X = (\mathbb{F}_q^n, +)$, with w_H the Hamming weight on X . Let $G = \text{Aut}(X, w_H)$ be the subgroup of $\text{Aut}(X)$ consisting of all linear automorphisms ϕ of X preserving the Hamming weight, i.e., $w_H(\phi u) = w_H(u)$, for all $u \in X$. Then, as is well-known, $G \cong S_n \times (\mathbb{F}_q^\times)^n$, where $\psi : S_n \rightarrow \text{Aut}((\mathbb{F}_q^\times)^n)$ is given by

$$\sigma \mapsto (\alpha = (\alpha_1, \dots, \alpha_n) \mapsto \alpha_{\sigma^{-1}} = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(n)})).$$

The isomorphism $S_n \times (\mathbb{F}_q^\times)^n \rightarrow G$ is given by

$$(\sigma, \alpha = (\alpha_1, \dots, \alpha_n)) \mapsto \phi_\sigma \phi_\alpha,$$

where

$$\phi_\sigma(x_1, \dots, x_n) = (x_{\sigma^{-1}1}, \dots, x_{\sigma^{-1}n}), \quad \phi_\alpha(x_1, \dots, x_n) = (\alpha_1 x_1, \dots, \alpha_n x_n).$$

Now, G acts naturally on X , and the G -orbits are $\mathcal{O}_i = \{x \in X | w_H(x) = i\}$ ($i = 0, 1, \dots, n$). Clearly, (3) and (4) are satisfied. The associated scheme \mathfrak{X}_G is nothing but the Hamming scheme $H(n, q)$. For the usual \mathbb{F}_q -valued inner product $x \cdot y = \sum_{i=1}^n x_i y_i$ on \mathbb{F}_q^n , we observe

$$\phi_\sigma \phi_\alpha x \cdot y = \sum_i \alpha_{\sigma^{-1}i} x_{\sigma^{-1}i} y_i = \sum_i x_i \alpha_i y_{\sigma i} = x \cdot \phi_{\sigma^{-1}} \phi_{\alpha_{\sigma^{-1}}} y.$$

So, for the inner product $\langle x, y \rangle = \lambda(x \cdot y)$, and $\iota : G \rightarrow G$ ($\phi_\sigma \phi_\alpha \mapsto \phi_{\sigma^{-1}} \phi_{\alpha_{\sigma^{-1}}}$), (8) is valid.

5 Further generalizations

The construction in Section 3 will be further generalized. Let X be a finite additive abelian group, and let G, \check{G} be two finite groups acting on X with their respective orbits $\mathcal{O}_0 = \{0\}, \mathcal{O}_1, \dots, \mathcal{O}_d$ and $\check{\mathcal{O}}_0 = \{0\}, \check{\mathcal{O}}_1, \dots, \check{\mathcal{O}}_d$. Assume both actions satisfy the conditions (3) and (4), with $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$, $\mathfrak{X}_{\check{G}} = (X, \{\check{R}_i\}_{i=0}^d)$ their respective associated schemes. As in (2), we let, for $0 \leq i \leq d$,

$$X_i = \{x \in X | (0, x) \in R_i\}, \quad \check{X}_i = \{x \in X | (0, x) \in \check{R}_i\},$$

so that

$$X_i = \mathcal{O}_i, \quad \check{X}_i = \check{\mathcal{O}}_i, \quad \text{for all } 0 \leq i \leq d.$$

A special case of the following theorem appeared in [11]. The proof is left to the reader, as it follows by slightly modifying that of Theorem 5. In the next theorem, $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}^\times$ is a fixed inner product.

Theorem 8 *The following are equivalent.*

- (a) $\mathfrak{X}_{\check{G}} = (X, \{\check{R}_i\}_{i=0}^d) \rightarrow \mathfrak{X}_G^* = (X^*, \{R_i^*\}_{i=0}^d)$, given by $x \mapsto \langle \cdot, x \rangle$, is an isomorphism of translation association schemes.
- (b) There is a permutation σ of $\{1, 2, \dots, d\}$ such that

$$x \in \check{X}_j \Leftrightarrow \langle \cdot, x \rangle \in X_{\sigma(j)}^*, \quad \text{for } j = 1, \dots, d.$$

- (c) $E_j = |X|^{-1} \sum_{x \in \check{X}_j} \langle \cdot, x \rangle^t \overline{\langle \cdot, x \rangle}$ ($0 \leq j \leq d$) are the irreducible idempotents

- of the association scheme $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$.
- (d) $f_j : X \rightarrow \mathbb{C}$, given by $f_j(y) = \sum_{x \in \check{X}_j} \langle y, x \rangle$ is constant on each X_i ($0 \leq i \leq d$), for all $j = 0, 1, \dots, d$.

Assume now further that there is a map $\iota : G \rightarrow \check{G}$ such that

$$\langle gx, y \rangle = \langle x, \iota(g)y \rangle, \text{ for all } g \in G, x, y \in X. \quad (9)$$

Then, as in Lemma 6, we have the following.

Lemma 9 *Under the assumption of (9), the sum*

$$\sum_{x \in \check{X}_j} \langle y, x \rangle \quad (y \in X_i)$$

depends only on i, j with $0 \leq i, j \leq d$.

So we obtain the following corollary from Theorem 8 and Theorem 1.

Corollary 10 *Let $\mathfrak{X}_G = (X, \{R_i\}_{i=0}^d)$ and $\mathfrak{X}_{\check{G}} = (X, \{\check{R}_i\}_{i=0}^d)$ be the two translation association schemes obtained from the actions of the finite groups G and \check{G} on the same finite abelian group X , and satisfying (3) and (4). Assume that there is a map $\iota : G \rightarrow \check{G}$ such that*

$$\langle gx, y \rangle = \langle x, \iota(g)y \rangle, \text{ for all } g \in G, x, y \in X,$$

where $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}^\times$ is an inner product. Then

- (a) $\mathfrak{X}_{\check{G}} = (X, \{\check{R}_i\}_{i=0}^d) \rightarrow \mathfrak{X}_G^* = (X^*, \{R_i^*\}_{i=0}^d)$ ($x \mapsto \langle \cdot, x \rangle$) is an isomorphism, i.e., \mathfrak{X}_G and $\mathfrak{X}_{\check{G}}$ are dual to each other,
- (b) $E_j = |X|^{-1} \sum_{x \in \check{X}_j} \langle \cdot, x \rangle {}^t \overline{\langle \cdot, x \rangle}$ ($0 \leq j \leq d$) are the irreducible idempotents for \mathfrak{X}_G ,
- (c) $q_{ij} = \sum_{x \in \check{X}_j} \langle y, x \rangle$ ($y \in X_i$) are the q -numbers for \mathfrak{X}_G ,
- (d) $p_{ij} = \sum_{x \in X_j} \langle y, x \rangle$ ($y \in \check{X}_i$) are the p -numbers for \mathfrak{X}_G ,
- (e) $p_{ij}^k = \check{q}_{ij}^k$, $p_{ij} = \check{q}_{ij}$, $m_i = \check{v}_i$, $q_{ij}^k = \check{p}_{ij}^k$, $q_{ij} = \check{p}_{ij}$, $v_i = \check{m}_i$.

6 An example for Section 5

Here we are content with giving only one example for Section 5 which is what we call the weak Hamming scheme. Let $H(m, q)$ denote, as usual, the Hamming scheme whose vertex set is \mathbb{F}_q^m and i -th relation is given by

$$(x, y) \in R_i \Leftrightarrow d_H(x, y) = w_H(x - y) = i \quad (i = 0, 1, \dots, m).$$

Here w_H and d_H are respectively the Hamming weight and the Hamming metric. Then the weak Hamming scheme $H(n_1, \dots, n_t, q)$ is given as the wreath product (cf. [1])

$$H(n_1, \dots, n_t, q) = H(n_1, q) \wr \dots \wr H(n_t, q),$$

so that the vertex set of that is $\mathbb{F}_q^{n_1} \times \dots \times \mathbb{F}_q^{n_t}$, and

$$(x, y) \in R_{n_1 + \dots + n_{i-1} + i_0} \\ \Leftrightarrow x_{i+1} = y_{i+1}, \dots, x_t = y_t \text{ and } d_H(x_i, y_i) = w_H(x_i - y_i) = i_0,$$

for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1$, $i_0 = 0$.

There is another description of $H(n_1, \dots, n_t, q)$ which has to do with poset-weight (poset-metric). This notion of poset-weight (poset-metric) was first introduced in [4]. Let $\mathbb{P} = ([n], \leq)$ be a poset, with $[n] = \{1, 2, \dots, n\}$. The \mathbb{P} -weight $w_{\mathbb{P}}$ is the function on \mathbb{F}_q^n , which is given by :

$$w_{\mathbb{P}}(x) = \#\{i \in [n] | i \leq j, \text{ for some } j \in \text{Supp}(x)\}.$$

Here $\text{Supp}(x) = \{j \in [n] | x_j \neq 0\}$, for $x = (x_1, \dots, x_n) \in \mathbb{F}_q^n$. Then $d_{\mathbb{P}}(x, y) = w_{\mathbb{P}}(x - y)$ is a distance function on \mathbb{F}_q^n , called \mathbb{P} -metric. We now specialize \mathbb{P} as the weak order poset \mathbb{P}_0 . $\mathbb{P}_0 = n_1 \mathbf{1} \oplus \dots \oplus n_t \mathbf{1}$ is given as the ordinal sum of the antichains $n_i \mathbf{1}$ on the set $\{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_{i-1} + n_i\}$, for $i = 1, \dots, t$, i.e., the underlying set is $[n]$ ($n = n_1 + \dots + n_t$) and the order relation is given by :

$$k < l \Leftrightarrow k \in n_i \mathbf{1}, l \in n_j \mathbf{1}, \text{ for some } i < j.$$

Let $G = \text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0})$ be the group of all linear automorphisms of \mathbb{F}_q^n preserving $w_{\mathbb{P}_0}$ -weight, i.e., $w_{\mathbb{P}_0}(\phi u) = w_{\mathbb{P}_0}(u)$, for all $u \in \mathbb{F}_q^n$. Then G acts on \mathbb{F}_q^n in a natural way and the G -orbits are $\mathcal{O}_i = \{x \in \mathbb{F}_q^n | w_{\mathbb{P}_0}(x) = i\}$, $i = 0, 1, \dots, n$. Clearly, the conditions (3) and (4) are valid. Moreover, for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1$, $i_0 = 0$,

$$y - x \in \mathcal{O}_{n_1 + \dots + n_{i-1} + i_0} \Leftrightarrow w_{\mathbb{P}_0}(y - x) = n_1 + \dots + n_{i-1} + i_0 \\ \Leftrightarrow x_{i+1} = y_{i+1}, \dots, x_t = y_t \text{ and } d_H(x_i, y_i) = i_0 \\ \Leftrightarrow (x, y) \in R_{n_1 + \dots + n_{i-1} + i_0}.$$

Here we identified \mathbb{F}_q^n with $\mathbb{F}_q^{n_1} \times \dots \times \mathbb{F}_q^{n_t}$ by writing the elements $x \in \mathbb{F}_q^n$ as the blocks of coordinates $x = (x_1, \dots, x_t) \in \mathbb{F}_q^{n_1} \times \dots \times \mathbb{F}_q^{n_t}$. So the associated scheme \mathfrak{X}_G is nothing but the weak Hamming scheme $H(n_1, \dots, n_t, q)$. Similarly, the weak Hamming scheme $H(n_t, \dots, n_1, q) = H(n_t, q) \wr H(n_{t-1}, q) \wr \dots \wr H(n_1, q)$ is also obtained from the action of the group $\tilde{G} = \text{Aut}(\mathbb{F}_q^n, w_{\tilde{\mathbb{P}}_0})$ on \mathbb{F}_q^n . Here $\tilde{\mathbb{P}}_0 = n_t \mathbf{1} \oplus n_{t-1} \mathbf{1} \oplus \dots \oplus n_1 \mathbf{1}$ is the dual poset of \mathbb{P}_0 .

Let $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, \dots, 0, 1)\}$ be the standard basis of \mathbb{F}_q^n . Let $g \in G = \text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0})$. Then, for each $1 \leq s \leq t$,

and each $i \in n_s \mathbf{1}$,

$$g(e_i) = a_i e_{\rho_s(i)} + \sum_{l=1}^{n_1+\dots+n_{s-1}} b_{li} e_l, \quad (10)$$

where $a_i \in \mathbb{F}_q^\times$, $b_{li} \in \mathbb{F}_q$, $\rho_s(i) \in n_s \mathbf{1}$, and the assignment $i \mapsto \rho_s(i)$ is a permutation on $n_s \mathbf{1}$. Conversely, if g is the linear map given by (10), for $1 \leq s \leq t$, $i \in n_s \mathbf{1}$, then $g \in G = \text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0})$. Let $x \cdot y = \sum_{i=1}^n x_i y_i$, for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{F}_q^n$. Then $\langle x, y \rangle = \lambda(x \cdot y)$ is an inner product on \mathbb{F}_q^n , where λ is a fixed nontrivial additive character on \mathbb{F}_q . Now, we will show the existence of a map $\iota : G = \text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0}) \rightarrow \check{G} = \text{Aut}(\mathbb{F}_q^n, w_{\check{\mathbb{P}}_0})$ such that $\langle gx, y \rangle = \langle x, \iota(g)y \rangle$, for all $g \in G$, $x, y \in \mathbb{F}_q^n$. For this, it is enough to show that there is a map $\iota : G \rightarrow \check{G}$ satisfying

$$ge_i \cdot e_k = e_i \cdot \iota(g)e_k, \text{ for all } g \in G, i, k = 1, \dots, n. \quad (11)$$

Let $g \in G$ be the map given by (10), for each $1 \leq s \leq t$, and each $i \in n_s \mathbf{1}$. If (11) is to be satisfied, we must have :

for $1 \leq j \leq t$, $k \in n_j \mathbf{1}$,

$$\begin{aligned} \iota(g)e_k \cdot e_i &= ge_i \cdot e_k \\ &= (a_i e_{\rho_s(i)} + \sum_{l=1}^{n_1+\dots+n_{s-1}} b_{li} e_l) \cdot e_k \\ &= \begin{cases} 0, & \text{if } s < j, \\ a_i, & \text{if } s = j \text{ and } k = \rho_s(i), \\ 0, & \text{if } s = j \text{ and } k \neq \rho_s(i), \\ b_{ki}, & \text{if } s > j. \end{cases} \end{aligned}$$

This yields that, for $1 \leq j \leq t$, $k \in n_j \mathbf{1}$,

$$\iota(g)e_k = a_{\rho_j^{-1}(k)} e_{\rho_j^{-1}(k)} + \sum_{l=n_1+\dots+n_j+1}^{n_1+\dots+n_t} b_{kl} e_l. \quad (12)$$

Now, $\iota(g)$ given by (12) belongs to $\check{G} = \text{Aut}(\mathbb{F}_q^n, w_{\check{\mathbb{P}}_0})$, so that a map $\iota : G \rightarrow \check{G}$ is defined and (11) is satisfied. Thus the results stated in Corollary 10 hold true. In particular, $H(n_1, n_2, \dots, n_t, q)$ and $H(n_t, n_{t-1}, \dots, n_1, q)$ are dual to each other.

Remark 11 (1) As we have seen in the above, the weak Hamming scheme $H(n_1, \dots, n_t, q)$ is the associated scheme \mathfrak{X}_G when $G = \text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0})$ ($n = n_1 + \dots + n_t$) acts on \mathbb{F}_q^n , with $\mathbb{P}_0 = n_1 \mathbf{1} \oplus \dots \oplus n_t \mathbf{1}$ the weak order poset. The weak order poset is unique in many respects. Indeed, the following has been shown. Let \mathbb{P} be a poset on $[n]$. Then (1) \mathbb{P} is a weak order poset on $[n] \Leftrightarrow$ (2) $(\mathbb{P}, \check{\mathbb{P}})$ is a weak dual MacWilliams pair (wdMp) \Leftrightarrow (3) The group $\text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}})$ acts transitively on each \mathbb{P} -sphere $S_{\mathbb{P}}(i) = \{x \in \mathbb{F}_q^n | w_{\mathbb{P}}(x) = i\}$ ($0 \leq i \leq n$) \Leftrightarrow (4) $\mathfrak{X} = (\mathbb{F}_q^n, \{R_i\}_{i=0}^n)$, with $(x, y) \in R_i \Leftrightarrow x - y \in S_{\mathbb{P}}(i)$ ($0 \leq i \leq n$), is

an association scheme. Here $\check{\mathbb{P}}$ is the dual poset of \mathbb{P} , a pair of posets $(\mathbb{P}, \check{\mathbb{P}})$ on $[n]$ is called a wdMp if the \mathbb{P} -weight distribution of C uniquely determines $\check{\mathbb{P}}$ -weight distribution of C^\perp , for every linear code $C \subseteq \mathbb{F}_q^n$. For details about these, one is referred to [9-11,13,14].

(2) The structure of $\text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0})$ was explicitly determined in [10]. So the map $\iota : G = \text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}_0}) \rightarrow \check{G} = \text{Aut}(\mathbb{F}_q^n, w_{\check{\mathbb{P}}_0})$ would have been more explicitly determined, just as we did in the example (g) of Section 4.

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